Kuwait University	**Math 102 (Calculus 2)**	**June 4, 2011**
Department of Mathematics	**Final Examination**	**Duration : 2 hours**

The use of calculators of all kinds is not allowed. All communication devices including mobile phones should be switched off. Answer all of the following questions.

1. Determine whether the statement is true or false. Justify your answer for credit. **(1 pt each)**

   (a) $\ln|f(x)|$ is a one-to-one function whenever $f(x)$ is a one-to-one function.

   (b) $\csc(\sin^{-1}(\frac{1}{x})) = x$ for all $|x| \geq 1$.

2. Let $f(x) = \sqrt{\sin^2 x + 2\sin x}$, $0 \leq x \leq \frac{\pi}{2}$.

   (a) Show that f is one-to-one. **(2 pts)**

   (b) Find the domain and range of f^{-1}, and compute $f^{-1}(x)$. **(4 pts)**

3. Find $\lim\limits_{x\to 0^+}\left(\coth x - \frac{1}{x}\right)$, if it exists. **(4 pts)**

4. Suppose f and f' are continuous on $[0,1]$, $f(1) = -1$, and $\int_0^1 f(x)dx = 2$. Find $\int_0^1 xf'(x)dx$. **(3 pts)**

5. Evaluate the following integrals. **(3 pts each)**

   (a) $\int (\sin x)^{-4}\sqrt{\tan x}\,dx$ (b) $\int \frac{\left(1-x^2\right)^{\frac{3}{2}}}{x}dx$

6. Find all $a > 0$ for which the integral $\int_0^\infty xa^{-x^2}dx$ is convergent, and find its value when it is convergent. **(4 pts)**

7. Find the centroid of the region bounded by the graphs of $y = \text{sech}^2 x$, $y = 0$, $x = 0$, and $x = \ln\sqrt{2}$. **(4 pts)**

8. Let C be the plane curve parameterized by

$$x = 2t - \sin 2t \qquad y = \cos 2t \qquad 0 \leq t \leq \frac{\pi}{4}$$

   (a) Find the length of C. **(3 pts)**

   (b) Find the area of the surface obtained by rotating C about the x-axis. **(3 pts)**

9. Let C be the polar curve $r = \sqrt{\theta}\cos\theta$, $0 \leq \theta \leq \frac{\pi}{2}$.

   (a) Sketch the curve. **(1 pt)**

   (b) Find the equation of the tangent line to C at the point corresponding to $\theta = \frac{\pi}{4}$. **(2 pts)**

   (c) Find the area enclosed by C. **(2 pts)**

1. (a) **Fals:** $f(x)=x$, $0<|x|<\infty$, $\ln|f(1)|=\ln|f(-1)|=0$

   (b) **True:** $|x|\geq 1 \Rightarrow 0<\left|\frac{1}{x}\right|\leq 1$, $\csc(\sin^{-1}(\frac{1}{x}))=\frac{1}{\sin(\sin^{-1}(\frac{1}{x}))}=\frac{1}{\frac{1}{x}}=x$

2. f is continuous and differentiable on $[0,\frac{\pi}{2}]$ with $f'(x)=\frac{(\sin x+1)\cos x}{\sqrt{\sin^2 x+2\sin x}}=\frac{(\sin x+1)\cos x}{\sqrt{(\sin x+2)\sin x}}>0$ for $0<x<\frac{\pi}{2}$.

   (a) f is increasing on $[0,\frac{\pi}{2}] \Rightarrow f$ is one-to-one.

   (b) $f(0)=0$, $f(\frac{\pi}{2})=\sqrt{3}$ $R_f=[0,\sqrt{3}]=D_{f^{-1}}$ $R_{f^{-1}}=D_f=[0,\frac{\pi}{2}]$
   Let $y\in[0,\sqrt{3}]$ and $0\leq x\leq\frac{\pi}{2}$ such that $f(x)=\sqrt{\sin^2 x+2\sin x}=y$. To solve for x, complete the square and get $\sin x=-1\pm\sqrt{1+y^2}$. But $0\leq x\leq\frac{\pi}{2}\Rightarrow \sin x\geq 0$, so the solution $-1-\sqrt{1+y^2}$ is rejected, and therefore $\sin x=-1+\sqrt{1+y^2}$. Since $x=\sin^{-1}\sin x$ $\forall\ 0\leq x\leq\frac{\pi}{2}$, we get $x=\sin^{-1}(\sqrt{1+y^2}-1)$. Therefore, $f^{-1}(x)=\sin^{-1}(\sqrt{1+x^2}-1)$.

3. $\coth x-\frac{1}{x}$ has the indeterminate form $\infty-\infty$ as $x\to 0^+$. Re-write it as

$\coth x-\frac{1}{x}=\frac{x\cosh x-\sinh x}{x\sinh x}$ and the last expression has the indeterminate form $\frac{0}{0}$ as $x\to 0^+$.

$\frac{(x\cosh x-\sinh x)'}{(x\sinh x)'}=\frac{x\sinh x}{\sinh x+x\cosh x}=\frac{\sinh x}{\frac{\sinh x}{x}+\cosh x}$; $\lim\limits_{x\to 0^+}\frac{\sinh x}{x}=\lim\limits_{x\to 0^+}\cosh x=1$ by L'Hospital's Rule

$\lim\limits_{x\to 0^+}\frac{(x\cosh x-\sinh x)'}{(x\sinh x)'}=\lim\limits_{x\to 0^+}\frac{\sinh x}{\frac{\sinh x}{x}+\cosh x}=\frac{0}{1+1}=\frac{0}{2}=0\Rightarrow \lim\limits_{x\to 0^+}\left(\coth x-\frac{1}{x}\right)=0$ by L'Hospital's Rule

4. Put $u=x$, $dv=f'(x)dx$ so that $du=dx$ and $v=f(x)$

$$\int_0^1 xf'(x)dx=\int_0^1 u\,dv=uv|_0^1-\int_0^1 v\,du=xf(x)|_0^1-\int_0^1 f(x)dx=-1-2=-3$$

5. (a) Write $\sin x=\cos x\tan x$ and replace $(\cos x)^{-1}$ by $\sec x$ to get

$$(\sin x)^{-4}\sqrt{\tan x}=(\cos x)^{-4}(\tan x)^{-4}(\tan x)^{\frac{1}{2}}=(\tan x)^{-\frac{7}{2}}\sec^4 x=(\tan x)^{-\frac{7}{2}}(1+\sec^2 x)\sec^2 x$$
$$\int(\sin x)^{-4}\sqrt{\tan x}dx=\int(1+u^2)u^{-\frac{7}{2}}du=-\tfrac{2}{5}u^{-\frac{5}{2}}-2u^{-\frac{1}{2}}+C,\quad u=\tan x,\quad -\tfrac{\pi}{2}<x<\tfrac{\pi}{2}$$
$$=\frac{-2}{5\sqrt{\tan x}\tan^2 x}-\frac{2}{\sqrt{\tan x}}+C$$

   (b) Use the trigonometric substitution $x=\sin\theta$, $0<|\theta|<\frac{\pi}{2}$, so that $dx=\cos\theta d\theta$. Then

$$\int\tfrac{1}{x}(1-x^2)^{\frac{3}{2}}dx=\int\frac{\cos^3\theta}{\sin\theta}\cos\theta d\theta=\int\frac{\left(1-\sin^2\theta\right)^2}{\sin\theta}d\theta=\int\frac{1-2\sin^2\theta+\sin^4\theta}{\sin\theta}d\theta$$
$$=\int\left(\csc\theta-2\sin\theta+\sin^3\theta\right)d\theta=\int\left(\csc\theta-2\sin\theta+(1-\cos^2\theta)\sin\theta\right)d\theta$$
$$=\ln|\csc\theta-\cot\theta|+2\cos\theta-\cos\theta+\tfrac{1}{3}\cos^3\theta+C$$
$$=\ln\left|\frac{1}{x}-\frac{\sqrt{1-x^2}}{x}\right|+2\sqrt{1-x^2}-\sqrt{1-x^2}+\tfrac{1}{3}(1-x^2)\sqrt{1-x^2}+C$$
$$=\ln\left|\frac{1-\sqrt{1-x^2}}{x}\right|+\tfrac{1}{3}\left(4-x^2\right)\sqrt{1-x^2}$$

6. $\int\limits_0^\infty xa^{-x^2}dx$ diverges for $0<a\leq 1$, and converges for $a>1$ and $\int\limits_0^\infty xa^{-x^2}dx=\frac{1}{2\ln a}$ in such case.

Reason: $\int_0^t xa^{-x^2}dx=\begin{cases}\frac{-1}{2\ln a}(a^{-t^2}-1), & \text{if } a\neq 1\\ \frac{t^2}{2}, & \text{if } a=1\end{cases}$, $\lim\limits_{t\to\infty}\frac{t^2}{2}=\infty$, $\lim\limits_{t\to\infty}a^{-t^2}=\begin{cases}0, & a>1\\ \infty, & 0<a<1\end{cases}$

7. $A = \text{Area} = \int\limits_0^{\ln\sqrt{2}} \operatorname{sech}^2 x dx = \frac{\sinh x}{\cosh x}\Big|_{\ln\sqrt{2}} = \frac{1}{3}$. If $C(\bar{x}, \bar{y})$ is the centroid, then

$$A\bar{x} = \int\limits_0^{\ln\sqrt{2}} x \operatorname{sech}^2 x dx = x \tanh x|_0^{\ln\sqrt{2}} - \int\limits_0^{\ln\sqrt{2}} \tanh x dx = x \tanh x|_0^{\ln\sqrt{2}} - \ln\cosh x|_0^{\ln\sqrt{2}} = \frac{5}{3}\ln 2 - \ln 3$$

$$\bar{x} = 5\ln 2 - 3\ln 3$$

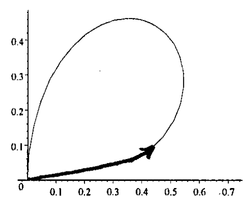
$$2A\bar{y} = \int\limits_0^{\ln\sqrt{2}} \operatorname{sech}^4 x dx = \int\limits_0^{\ln\sqrt{2}} (1 - \tanh^2 x) \operatorname{sech}^2 x dx = [\tanh x - \tfrac{1}{3}\tanh^3 x]_0^{\ln\sqrt{2}} = \frac{26}{81} \Rightarrow \bar{y} = \frac{39}{81}$$

8. We have $\frac{dx}{dt} = 2(1 - \cos 2t)$, $\frac{dy}{dt} = -2\sin 2t$, $\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} = 2\sqrt{2}\sqrt{1 - \cos 2t} = 4|\sin t|$

(a) $L = 2\sqrt{2}\int_0^{\frac{\pi}{4}} \sqrt{1 - \cos 2t} dt = 4\int_0^{\frac{\pi}{4}} (\sin t) dt = -4(\frac{1}{\sqrt{2}} - 1) = 2(2 - \sqrt{2})$

(b) $S = 4\sqrt{2}\pi\int_0^{\frac{\pi}{4}} (\cos 2t)\sqrt{1 - \cos 2t} dt = 8\pi\int_0^{\frac{\pi}{4}} (\cos 2t)(\sin t) dt = 8\pi\int_0^{\frac{\pi}{4}} (2\cos^2 t - 1)(\sin t) dt$
$= 8\pi\int_0^{\frac{\pi}{4}} (2\cos^2 t \sin t - \sin t) dt = 8\pi(-\frac{2}{3}\cos^3 t + \cos t|_0^{\frac{\pi}{4}}) = \frac{8\pi}{3}(\sqrt{2} - 1)$

9. The sketch is below.



Parameterize C with the equations: $x = r\cos\theta = \sqrt{\theta}\cos^2\theta$, $y = r\sin\theta = \sqrt{\theta}\cos\theta\sin\theta$, $0 \le \theta \le \frac{\pi}{2}$.
Then

$$\frac{dy}{dx} = \frac{\frac{d}{d\theta}(\sqrt{\theta}\cos\theta\sin\theta)}{\frac{d}{d\theta}(\sqrt{\theta}\cos^2\theta)} = \frac{\frac{1}{2}(\frac{1}{2\sqrt{\theta}}\sin 2\theta + 2\sqrt{\theta}\cos 2\theta)}{\frac{1}{2\sqrt{\theta}}\cos^2\theta - \sqrt{\theta}\sin 2\theta} = \frac{\sin 2\theta + 4\theta\cos 2\theta}{2\cos^2\theta - 4\theta\sin 2\theta} \Rightarrow \frac{dy}{dx}\Big|_{\frac{\pi}{4}} = \frac{1}{1-\pi}$$

$x = \frac{1}{4}\sqrt{\pi}$, $y = \frac{1}{4}\sqrt{\pi}$ are the x and y coordinates of the tangency point

$y = \frac{1}{1-\pi}(x - \frac{1}{4}\sqrt{\pi}) + \frac{1}{4}\sqrt{\pi}$ or $4x + 4(\pi - 1)y + \pi\sqrt{\pi} = 0$ is the equation of the tangent line

$$\text{Area} = \frac{1}{2}\int\limits_0^{\frac{\pi}{2}} \theta\cos^2\theta d\theta = \frac{1}{4}\int\limits_0^{\frac{\pi}{2}} \theta(1 + \cos 2\theta) d\theta = \frac{1}{4}\int\limits_0^{\frac{\pi}{2}} u dv \qquad \text{by parts: } u = \theta,\ dv = (1 + \cos 2\theta) d\theta$$

$$= \frac{1}{4}[\theta(\theta + \tfrac{1}{2}\sin 2\theta)|_0^{\frac{\pi}{2}} - \int\limits_0^{\frac{\pi}{2}} (\theta + \tfrac{1}{2}\sin 2\theta) d\theta] = \frac{1}{4}[\frac{\pi^2}{4} - (\tfrac{1}{2}\theta^2 - \tfrac{1}{4}\cos 2\theta)|_0^{\frac{\pi}{2}} = \frac{1}{32}(\pi^2 - 4)$$